$$\frac{e^{F_{k+1}}-e^{F_k}}{F_{k+1}-F_k}=\frac{e^{F_{k+1}}-e^{F_k}}{F_{k-1}}\geq e^{\frac{F_k+F_{k+1}}{2}}=e^{\frac{F_{k+2}}{2}}$$

so

$$\prod_{k=1}^{n}\frac{e^{F_{k+1}}-e^{F_k}}{F_{k-1}}\geq\prod_{k=1}^{n}e^{\frac{F_{k+2}}{2}}=e^{\frac{1}{2}\sum\limits_{k=1}^{n}F_{k+2}}=e^{\frac{1}{2}(F_{n+4}-5)}$$

Mihály Bencze

Second solution. First we will prove that for any $x,y\in\mathbb{R}$ such that $x\neq y$ holds inequality

$$\frac{e^x-e^y}{x-y}\geq e^{\frac{x+y}{2}}. \tag{1}$$

Without loss of generality we can assume that $x>y$ (because $\dfrac{e^x-e^y}{x-y}=\dfrac{e^y-e^x}{y-x}$).
Then

$$\frac{e^x-e^y}{x-y}\geq e^{\frac{x+y}{2}}\iff\frac{e^{\frac{x-y}{2}}-e^{\frac{y-x}{2}}}{2\cdot\dfrac{x-y}{2}}\geq 1\iff\sinh\left(\frac{x-y}{2}\right)\geq\frac{x-y}{2}.$$

Latter inequality holds because $\sinh t\geq t$ for any real $t\geq 0$.
Indeed since $(\sinh t-t)'=\cosh t-1\geq 0$ then $\sinh t-t$ increasing in $[0,\infty)$ and, therefore,

$$\sinh t-t\geq\sinh 0-0=0.$$

Noting that $F_{k-1}=F_{k+1}-F_k$ and using inequality (1) we obtain

$$\prod_{k=1}^{n}\frac{e^{F_{k+1}}-e^{F_k}}{F_{k-1}}=\prod_{k=1}^{n}\frac{e^{F_{k+1}}-e^{F_k}}{F_{k+1}-F_k}\geq\prod_{k=1}^{n}e^{\frac{F_{k+1}+F_k}{2}}=\prod_{k=1}^{n}e^{\frac{F_{k+2}}{2}}=e^{\frac{1}{2}\sum_{k=1}^{n}F_{k+2}}.$$

Since $\sum_{k=1}^{n}F_{k+2}=\sum_{k=1}^{n}(F_{k+4}-F_{k+3})=F_{n+4}-F_4$ and $F_4=5$ then

$$\prod_{k=1}^{n} \frac{e^{F_{k+1}} - e^{F_k}}{F_{k-1}} \geq e^{\frac{1}{2}(F_{n+4}-5)}.$$

Arkady Alt

W54. (Solution by the proposer.) We have the followings:

$$\sum \frac{a_1}{x^2+a_2} = \sum \frac{a_1^2}{a_1x^2+a_1a_2} \geq \frac{\left(\sum a_1\right)^2}{\sum\left(a_1x^2+a_1a_2\right)} = \frac{\sum a_1}{x^2+\frac{\sum a_1a_2}{\sum a_1}}$$

and

$$\int_0^1 \frac{a_1}{x^2+a_2}dx \geq \int_0^1 \frac{\sum a_1}{x^2+\frac{\sum a_1a_2}{\sum a_1}}dx \text{ or } \sum \frac{a_1}{\sqrt{a_2}} arctg \frac{x}{\sqrt{a_2}}\Bigg|_0^1 \geq$$

$$\geq \frac{\left(\sum a_1\right)^{\frac{3}{2}}}{\sqrt{\sum a_1a_2}} arctg x\sqrt{\frac{\sum a_1}{\sum a_1a_2}}\Bigg|_0^1$$

and finally

$$\sum \frac{a_1}{\sqrt{a_2}} arctg \frac{1}{\sqrt{2}} \geq \frac{\left(\sum a_1\right)^{\frac{3}{2}}}{\sqrt{\sum a_1a_2}} arctg\sqrt{\frac{\sum a_1}{\sum a_1a_2}}$$

Mihály Bencze

Second solution. Let

$$f(x) = \frac{1}{\sqrt{x}} \tan^{-1} \frac{1}{\sqrt{x}}$$

for all $x > 0$. Now,

$$f'(x) = -2\tan^{-1}\frac{1}{\sqrt{x}}\left(\frac{1}{\sqrt{x^3}}+\frac{1}{x^2}\right)$$

and

$$f''(x) = \frac{4}{\sqrt{x^3}}\tan^{-1}\frac{1}{\sqrt{x}}\left(\frac{1}{\sqrt{x^3}}+\frac{1}{x^2}\right)+2\tan^{-1}\frac{1}{\sqrt{x}}\left(\frac{3}{2}\frac{1}{\sqrt{x^5}}+\frac{2}{x^3}\right) > 0$$

for all $x > 0$, therefore f is convex for all $x > 0$ again